Columnar Phase in Quantum Dimer Models

Alessandro Giuliani[1] and Elliott H. Lieb[2]

[1]Dipartimento di Matematica e Fisica, Università di Roma Tre
L.go S. L. Murialdo 1, 00146 Roma, Italy

[2]Departments of Mathematics and Physics, Jadwin Hall, Princeton University
Princeton 08544 NJ, USA

April 29, 2015

Abstract

The quantum dimer model, relevant for short-range resonant valence bond physics, is rigorously shown to have long range order in a crystalline phase in the attractive case at low temperature and not too large flipping term. This term flips horizontal dimer pairs to vertical pairs (and vice versa) and is responsible for the word 'quantum' in the title. In addition to the dimers, monomers are also allowed. The mathematical method used is 'reflection positivity'. The model and proof can easily be generalized to dimers or plaquettes in 3-dimensions.

1 Introduction

Starting with the work of Rokhsar and Kivelson [17] quantum dimer models became popular in connection with the study of resonating valence bond (RVB) states, as well as in the study of cold bosons, frustrated magnetism, Josephson junction arrays, and other physical models, see [13] for a review. In these models dimers on a square lattice not only have a pair interaction that favors or disfavors parallel dimers next to each other, but they also have

a flipping term that exchanges a horizontal pair with a vertical pair. It is this 'non-diagonal' flipping term that gives rise to the appellation 'quantum'.

Many kinds of physical phases can arise from such models. As the interaction is increased from large negative values (strongly attractive case) to large positive values (strongly repulsive case), quantum dimers are expected to exhibit a remarkable series of transitions: for instance, on the square lattice, the system is expected to pass from a crystalline "columnar" state, to a "mixed" state, then a "plaquette", and finally a "staggered" state, passing through an anomalous liquid phase at the Rokshar-Kivelson point, see [13, 16] for details about these phases and transitions. Up to now, none of these conjectured phases and transitions has been rigorously established. For recent rigorous results about the anomalous liquid phase in the *classical* interacting dimer model, see [8, 9].

Here we study the attractive case on the square lattice and rigorously establish that there is a columnar state at low temperature provided the monomer density and flipping rate are not too large. Our analysis easily extends to the three-dimensional cubic lattice, in which case the dimers can remain as dimers or they can be replaced by two-dimensional plaquettes.

Much before Ref.[17], Heilmann and Praestgaard [10, 11, 12], using a Peierls contour argument, proved that the same model, but without the flipping term, has columnar long-range order (LRO) at low temperature and not too large monomer density.

The introduction of the quantum feature complicates the proof of LRO considerably. Borgs et al and Datta et al [1, 2] proved, quite generally, that quantum perturbations of classical models do not destroy LRO at low temperature. Their methods are both based on a quantum extension of Pirogov-Sinai theory [15].

In our case the method we use – reflection positivity (RP) and chessboard estimates – is much simpler. While it was introduced in quantum field theory by Osterwalder and Schrader [14], its first application in classical statistical mechanics goes back to Fröhlich, Simon and Spencer [7], and to Dyson, Lieb and Simon for the quantum case [3]. See also [6] for an application to the anisotropic quantum Heisenberg models, which is relevant for our proof here. The use of RP has added advantages apart from brevity. When applicable it probably gives better estimates on the range in which LRO holds. In (2.2) we give an explicit sufficient condition for LRO. It also establishes some correlation positivity results, which can be used for some correlation inequalities.

2 The model

We introduce a quantum dimer model on the 2D square lattice (which can be easily extended to 3D): consider a square portion of $\mathbb{Z}^2$ of even side L with periodic boundary conditions (for convenience later we will choose L to be divisible by 4). The Hamiltonian is conventionally expressed in the following way:

$$H_\Lambda = -\varepsilon \sum_{P\subset\Lambda} (v_P^\dagger h_P + h_P^\dagger v_P) - \mu \sum_{P\subset\Lambda} (v_P^\dagger v_P + h_P^\dagger h_P) + zM \equiv -\varepsilon T - \mu U + zM, \tag{2.1}$$

where the summations run over the plaquettes P of Λ. The physical objects on the lattice are dimers, which lie on the edges, and the symbols h_P (resp. v_P) denote operators that kill two horizontal (resp. vertical) dimers occupying P. Correspondingly, $h_P^\dagger$ and $v_P^\dagger$ create pairs of parallel dimers. Of course P can have no dimers, or one dimer, in which case h_P and v_P annihilate the corresponding state; this means that $h_P|\phi\rangle = v_P|\phi\rangle = 0$ if $|\phi\rangle$ is a classical dimer configuration such that P has zero or one dimers. We *do not* assume complete dimer covering: if N_v and N_h denote the number of dimers in the vertical and horizontal directions, $M = |\Lambda| - 2N_v - 2N_h$ denotes the number of monomers (empty sites).

The first term in (2.1) switches the dimers in a doubly occupied plaquette, while the second simply counts the number of such plaquettes. 3D extensions of the model, treatable by our methods, are, e.g., the following: (i) same Hamiltonian in 3D, where now plaquettes can be in three possible directions; (ii) same thing, but with the interaction term replaced by a similar one, counting the number of unit cubes completely filled by four parallel dimers; (iii) quantum Hamiltonian for plaquettes (rather than dimers): parallel neighboring plaquettes interact and can flip whenever they occupy the same unit cube. See section 3.

The sign of the constant ε is conventional, because it can be changed by the unitary transformation i^{N_v}. Therefore, we will assume that $\varepsilon > 0$. The sign of the second term is very important: we shall be concerned with the case that $\mu > 0$. By rescaling, we assume $\mu = 1$. In these units, we assume that the monomer chemical potential, z, is larger than $-1/2$.

The Hamiltonian preserves the number of dimers (or monomers) and this divides the Hilbert space into sectors, each of which has a spectrum. At $\varepsilon = 0$, there are four ground states, called columnar states, and these are

in the sector where the number of dimers is $|\Lambda|/2$. One of them consists of parallel columns of horizontal dimers, and the other three are obtained by shifting the columns one lattice space to the right, or else rotating the whole picture(s) by 90^o.

As anticipated above, we prove that columnar LRO, in the sense defined precisely in (2.10), is present at small enough $\varepsilon > 0$ and large enough β, by using quantum reflection positivity, chessboard estimates, exponential localization and an adaptation of the classical Peierls argument.

More exactly, we prove that a sufficient condition for LRO is that the following inequality is satisfied by the three parameters β, ε and z:

$$\boxed{\sum_{n\geq 2} n^2 \alpha^n < \frac{3}{7} \quad \text{with} \quad \alpha = 18 \max\Big\{ e^{-\beta(\frac{1}{16}-\frac{\varepsilon}{2})} + \big(\frac{2\varepsilon}{1/16-\varepsilon}\big)^{\frac{1}{24}},\, e^{-\frac{1}{2}\beta(z+1/2)} \Big\}} \tag{2.2}$$

2.1 Reflection positivity

We first show that the Hamiltonian (2.1) in a suitable representation is reflection positive. We associate with each point $\mathbf{x} \in \Lambda$ a Hilbert space $\mathcal{H}_{\mathbf{x}} = \mathbb{C}^5$; there are 5 allowed states at $\mathbf{x}$, one of them is the monomer, and the other 4 refer to the half-dimers pointing in the directions north, east, south, west. The total Hilbert space is the tensor product of these local spaces. To ensure that we do not have unpaired half-dimers, we will add to our Hamiltonian a term that suppresses unpaired half-dimers. At each point we introduce the operators $\mathbb{1}^{\#}_{\mathbf{x}}$, with $\# \in \{0, N, E, S, W\}$, which are projection operators onto the 5 different states, the rotation operator $R^{N\to E}_{\mathbf{x}}$, which flips the state N into E and gives zero otherwise, and similarly $R^{N\to W}_{\mathbf{x}}$, etc.

Consider the Hamiltonian:

$$\begin{aligned} H^{\lambda}_{\Lambda} \;=\; & \lambda\Big[|\Lambda| - 2\sum_{\mathbf{x}} (\mathbb{1}^{E}_{\mathbf{x}}\mathbb{1}^{W}_{\mathbf{x}+\hat{e}_1} + \mathbb{1}^{N}_{\mathbf{x}}\mathbb{1}^{S}_{\mathbf{x}+\hat{e}_2}) - \sum_{\mathbf{x}} \mathbb{1}^{0}_{\mathbf{x}}\Big] + z\sum_{\mathbf{x}} \mathbb{1}^{0}_{\mathbf{x}} && (2.3)\\ & -\varepsilon \sum_{P=(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)} (R^{E\to N}_{\mathbf{x}_1} R^{W\to N}_{\mathbf{x}_2} R^{W\to S}_{\mathbf{x}_3} R^{E\to S}_{\mathbf{x}_4} + h.c.) && (2.4)\\ & - \sum_{P=(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)} (\mathbb{1}^{N}_{\mathbf{x}_1}\mathbb{1}^{N}_{\mathbf{x}_2}\mathbb{1}^{S}_{\mathbf{x}_3}\mathbb{1}^{S}_{\mathbf{x}_4} + \mathbb{1}^{E}_{\mathbf{x}_1}\mathbb{1}^{W}_{\mathbf{x}_2}\mathbb{1}^{W}_{\mathbf{x}_3}\mathbb{1}^{E}_{\mathbf{x}_4}) && (2.5) \end{aligned}$$

where the sums in the last two lines runs over the plaquettes of Λ, each plaquette P being thought as the union of the four sites $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ at the

left bottom, right bottom, right top, left top of P, respectively. As we let $\lambda \to \infty$, H_Λ^∞ gives infinite energy to unphysical configurations (i.e., configurations with unpaired half-dimers) and on the physical space it becomes equivalent to (2.1) at $\mu = 1$: note, in fact, that the term in square brackets on the first line is zero on physical configurations and positive otherwise; the remaining terms (2.4) and (2.5) reduce on the physical space to the first and second terms in (2.1), and similarly $z \sum_{\mathbf{x}} \mathbb{1}_{\mathbf{x}}^0$ reduces to the last term in (2.1).

In this representation, there is a natural notion of reflection positivity. Take a pair of planes passing through the bonds and cutting Λ in two equal sized halves, Λ_- and Λ_+. Reflection about these planes consists in changing the site labels into their reflected images, and exchanging up with down (N with S) or right with left (E with W), depending on whether the planes are horizontal or vertical. We call θ such a reflection; it satisfies the conditions listed in [5]. Given any pair of reflection planes, we can write $-H_\Lambda^\lambda$ as $A + \theta A + \sum_i C_i \theta C_i$, with A representing the restriction of $-H_\Lambda^\lambda$ to one of the two halves, say Λ_+, and $C_i \theta C_i$ being the terms connecting the two halves. Therefore, H_Λ^λ is Reflection Positive (RP), which follows from the general theory in [5, Sections 2 and 3.3]. This means that if we take any operator F supported on Λ_+ then

$$\left\langle F^\dagger (\theta F) \right\rangle_\lambda \geq 0, \tag{2.6}$$

where $\langle (\cdot) \rangle_\lambda = \mathrm{Tr} e^{-\beta H_\Lambda^\lambda} (\cdot) / Z_\Lambda^\lambda$, with $Z_\Lambda^\lambda = \mathrm{Tr} e^{-\beta H_\Lambda^\lambda}$. Moreover, the chessboard estimate holds, see [5, Theorem 4.1]. If, e.g., $\{f_{\mathbf{x}}\}_{\mathbf{x} \in \Lambda}$ are operators supported on the sites of Λ

$$\left\langle \prod_{\mathbf{x} \in \Lambda} f_{\mathbf{x}} \right\rangle_\lambda \leq \prod_{\mathbf{x} \in \Lambda} \left[\langle F_{\mathbf{x}} \rangle_\lambda \right]^{1/|\Lambda|}, \tag{2.7}$$

where $F_{\mathbf{x}} = \prod_{\mathbf{z} \in \Lambda} \tilde{f}_{\mathbf{x}}^{(\mathbf{z})}$, and $\tilde{f}_{\mathbf{x}}^{(\mathbf{z})}$ is a copy of $f_{\mathbf{x}}$ attached to the site $\mathbf{z}$ if $\mathbf{z}$ and $\mathbf{x}$ belong to the same colored sublattice on the chessboard. If they belong to differently colored sublattices, then $\tilde{f}_{\mathbf{x}}^{(\mathbf{z})}$ is a copy of $\theta f_{\mathbf{x}}$ attached to the site $\mathbf{z}$. Eq.(2.7) is obtained by repeatedly reflecting the left side about planes not passing through the sites.

Similar considerations apply to the case in which we have to evaluate the average of a product of operators supported on the occupied bonds of one of the four classical (columnar) ground states. In this case we repeatedly reflect about planes that pass neither through sites, nor through the occupied bonds. For these kind of reflections, we need L to be divisible by 4. In the next section we will apply this analogue of (2.7) to estimate the probability

of any Peierls contour in terms of that of the 'universal Peierls contour' that is as large as Λ itself, and is much easier to evaluate.

Note that (2.6) and (2.7) remain valid as $\lambda \to \infty$, in which case the average $\langle\cdot\rangle_\lambda$ reduces to the Gibbs measure of the original quantum dimer model, simply denoted by $\langle\cdot\rangle$. From now on we restrict ourselves to this limiting case, and we implicitly assume that no unphysical configuration of half-dimers appears in the states under consideration. We also drop the ∞ label from H_Λ^∞ and identify H_Λ with $-U - \varepsilon T + zM$, as in (2.1) (recall that we fixed $\mu = 1$).

2.2 The Peierls argument

We introduce on-site projectors associated with the 4 classical ($\varepsilon = 0$) ground states, labelled $1, 2, 3, 4$, as follows:

$$\mathbb{1}_{\mathbf{x}}^1 = \begin{cases} \mathbb{1}_{\mathbf{x}}^E \text{ if } x_1 \text{ is even} \\ \mathbb{1}_{\mathbf{x}}^W \text{ if } x_1 \text{ is odd} \end{cases}, \qquad \mathbb{1}_{\mathbf{x}}^2 = \begin{cases} \mathbb{1}_{\mathbf{x}}^W \text{ if } x_1 \text{ is even} \\ \mathbb{1}_{\mathbf{x}}^E \text{ if } x_1 \text{ is odd} \end{cases}, \tag{2.8}$$

$$\mathbb{1}_{\mathbf{x}}^3 = \begin{cases} \mathbb{1}_{\mathbf{x}}^N \text{ if } x_2 \text{ is even} \\ \mathbb{1}_{\mathbf{x}}^S \text{ if } x_2 \text{ is odd} \end{cases}, \qquad \mathbb{1}_{\mathbf{x}}^4 = \begin{cases} \mathbb{1}_{\mathbf{x}}^S \text{ if } x_2 \text{ is even} \\ \mathbb{1}_{\mathbf{x}}^N \text{ if } x_2 \text{ is odd} \end{cases}. \tag{2.9}$$

We want to prove Long Range Order (LRO), in the sense that

$$\sum_{i=1}^4 \left\langle \mathbb{1}_{\mathbf{x}}^i \mathbb{1}_{\mathbf{y}}^i \right\rangle > 1/4, \tag{2.10}$$

for all $\Lambda, \mathbf{x}, \mathbf{y}$. The condition (2.10) guarantees that the infinite volume Gibbs state $\langle\cdot\rangle_{\mathbb{Z}^2} = \lim_{|\Lambda|\to\infty} \langle\cdot\rangle$ is not pure, i.e., it does not satisfy the clustering property[1]: therefore, (2.10) implies that the model exhibits a phase transition, in that it admits multiple pure states (at least four, by the translational and rotational symmetries). To see that (2.10) violates the clustering property, note that $\langle \mathbb{1}_{\mathbf{x}}^i \rangle = (1-p_0)/4$, $\forall \mathbf{x} \in \Lambda, \forall i = 1, 2, 3, 4$, where $p_0 = \langle \mathbb{1}_{\mathbf{x}}^0 \rangle$. The clustering property would imply $\lim_{|\mathbf{x}-\mathbf{y}|\to\infty} \sum_{i=1}^4 \left\langle \mathbb{1}_{\mathbf{x}}^i \mathbb{1}_{\mathbf{y}}^i \right\rangle = (1-p_0)^2/4 \leq 1/4$, which is in contradiction with (2.10).

[1] The clustering property is the condition that $\lim_{|\mathbf{x}-\mathbf{y}|\to\infty} \langle A_{\mathbf{x}} B_{\mathbf{y}} \rangle_{\mathbb{Z}^2} = \langle A_{\mathbf{x}} \rangle_{\mathbb{Z}^2} \langle B_{\mathbf{y}} \rangle_{\mathbb{Z}^2}$ for any two observables $A_{\mathbf{x}}$ and $B_{\mathbf{y}}$ supported around $\mathbf{x}$ and $\mathbf{y}$, respectively.

Recalling that $\sum_{i,j=0}^{4} \langle \mathbb{1}_{\mathbf{x}}^i \mathbb{1}_{\mathbf{y}}^j \rangle = 1$, a sufficient condition for (2.10) is that for small enough temperature and monomer density,

$$\langle \mathbb{1}_{\mathbf{x}}^i \mathbb{1}_{\mathbf{y}}^j \rangle < \tfrac{1}{21} \cdot \tfrac{3}{4} = \tfrac{1}{28}, \quad \text{if } 0 < i, j \leq 4 \text{ and } i \neq j, \tag{2.11}$$

$$\langle \mathbb{1}_{\mathbf{x}}^0 \mathbb{1}_{\mathbf{y}}^i \rangle \leq \langle \mathbb{1}_{\mathbf{x}}^0 \rangle < \tfrac{1}{28}, \quad \forall i = 0, \dots, 4, \tag{2.12}$$

for all $\Lambda, \mathbf{x}, \mathbf{y}$. The number $21 = 25 - 4$ is the number of possible 'bad pairs' of indices attached to the projectors in $\mathbf{x}$ and $\mathbf{y}$.

In order to bound the left side of (2.11), we insert at each site $\mathbf{z}$ different from $\mathbf{x}$ and $\mathbf{y}$ a resolution of the identity $I = \sum_{i=0}^{4} \mathbb{1}_{\mathbf{z}}^i$, and then we expand the product of sums, thus getting a sum over *configurations* of projectors (we repeat that, since we let $\lambda \to \infty$, only physical configurations are retained). These configurations are the quantum analogues of the classical dimer configurations in [11]. In analogy with the classical case, we associate each configuration with a set of Peierls contours [10, 11, 12], which are closed paths on the dual lattice, separating either good regions of different types (a good region being a connected region covered by projectors all with the same index $i \in \{1, 2, 3, 4\}$) or a good region from a bad region. We identify the contour enclosing the good region containing $\mathbf{x}$ and call it γ. Every bond *crossing* γ comes with two projectors associated with a bad pair of indices. These bonds can be of 4 different types, depending on whether they are of the form $(\mathbf{x}, \mathbf{x} + \hat{e}_1)$ with x_1 even, $(\mathbf{x}, \mathbf{x} + \hat{e}_1)$ with x_1 odd, $(\mathbf{x}, \mathbf{x} + \hat{e}_2)$ with x_2 even, or $(\mathbf{x}, \mathbf{x} + \hat{e}_2)$ with x_2 odd. We single out the most numerous among these four types, which consist, therefore, of at least $|\gamma|/4$ elements; let T_γ be the corresponding set of bonds.

For the purpose of an upper bound, we retain only the projectors on the bonds in T_γ and throw away (i.e., bound by 1) all the others. As a result,

$$\langle \mathbb{1}_{\mathbf{x}}^i \mathbb{1}_{\mathbf{y}}^j \rangle \leq \sum_{\gamma \succ \mathbf{x}} \sum_{\{(i_{\mathbf{z}}, i_{\mathbf{z}'})\}_{(\mathbf{z},\mathbf{z}') \in T_\gamma}} \Big\langle \prod_{(\mathbf{z},\mathbf{z}') \in T_\gamma} P_{\mathbf{z}}^{i_{\mathbf{z}}} P_{\mathbf{z}'}^{i_{\mathbf{z}'}} \Big\rangle, \tag{2.13}$$

where the first sum runs over contours enclosing $\mathbf{x}$ and the second over the bad pairs of indices associated with the bonds $(\mathbf{z}, \mathbf{z}')$ in T_γ; if $\mathbf{z}$ is the site belonging to the same good region as $\mathbf{x}$, then $i_{\mathbf{z}} = i$.

In order to evaluate $\langle \prod_{(\mathbf{z},\mathbf{z}') \in T_\gamma} P_{\mathbf{z}}^{i_{\mathbf{z}}} P_{\mathbf{z}'}^{i_{\mathbf{z}'}} \rangle$, we use the chessboard estimate [5, Theorem 4.1], in the way explained at the end of previous section, thus finding

$$\Big\langle \prod_{(\mathbf{z},\mathbf{z}') \in T_\gamma} P_{\mathbf{z}}^{i_{\mathbf{z}}} P_{\mathbf{z}'}^{i_{\mathbf{z}'}} \Big\rangle \leq \delta^{|T_\gamma|}, \tag{2.14}$$

where

$$\delta^{|\Lambda|/2} = \max\left\{\left\langle P^{34}\right\rangle, \left\langle P^{32}\right\rangle, \left\langle P^{30}\right\rangle, \left\langle P^{20}\right\rangle\right\} \tag{2.15}$$

and

$$P^{ij} = \prod_{\mathbf{x}:\ x_1\in 4\mathbb{Z}} \mathbb{1}^i_{\mathbf{x}}\mathbb{1}^j_{\mathbf{x}+\hat{e}_1}\mathbb{1}^j_{\mathbf{x}+2\hat{e}_1}\mathbb{1}^i_{\mathbf{x}+3\hat{e}_1} \tag{2.16}$$

is a *universal projector*, in the sense of [6], onto a particular periodic classical state. The cases of interest, that is P^{34}, P^{32}, P^{30} and P^{20}, are described in Figure 1. Each one involves all the vertices in Λ, not just those in the bonds of T_γ.

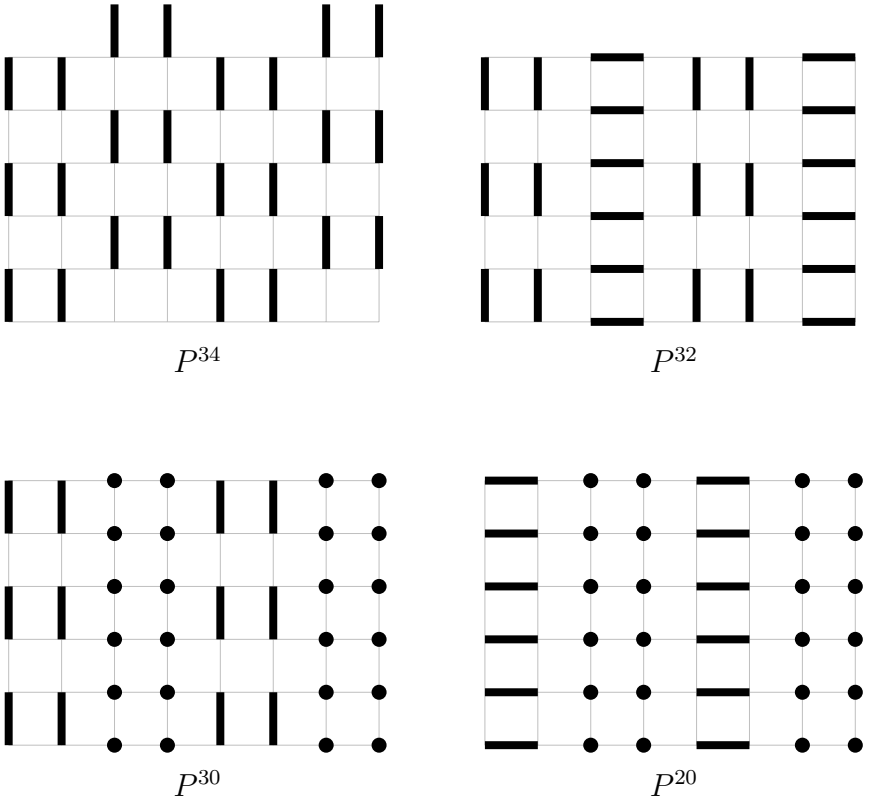

Figure 1: A pictorial representation of the four universal projectors that are obtained after repeated reflections of the bad bonds crossing the contours. See (2.14) *et seq.* In the third and fourth pictures the dots represent monomers.

Similarly, the left side of (2.12) can be bounded by $\langle P^0\rangle^{1/|\Lambda|}$, with $P^0 = \prod_{\mathbf{x}\in\Lambda} \mathbb{1}^0_{\mathbf{x}}$. For later use, we also define $P^i = \prod_{\mathbf{x}\in\Lambda} \mathbb{1}^i_{\mathbf{x}}$.

We will prove that

$$\big\langle P^0\big\rangle^{1/|\Lambda|} \leq e^{-\beta(z+1/2)}, \tag{2.17}$$

which implies (2.12) for $z > -1/2$ and β large enough. Moreover,

$$\delta \leq \max\Big\{e^{-\beta(1/8-\varepsilon)} + \big(\frac{2\varepsilon}{1/16-\varepsilon}\big)^{1/12}, e^{-\beta(z+1/2)}\Big\} \tag{2.18}$$

which is as small as desired, provided $z > -1/2$, and ε and β^{-1} are sufficiently small.

Assuming this, the validity of (2.11) and, therefore, existence of LRO, follows from the standard Peierls estimate: by (2.13), (2.14), we get that if $i, j > 0$ and $i \neq j$, and δ is small enough,

$$\big\langle \mathbb{1}^i_{\mathbf{x}} \mathbb{1}^j_{\mathbf{y}}\big\rangle \leq \sum_{\gamma \succ \mathbf{x}} (4\delta)^{|\gamma|/4} \tag{2.19}$$

Now, the number of contours of length $2n$ (note that the length of each contour is even) enclosing $\mathbf{x}$ is smaller than $n^2 9^n/12$, which gives $\big\langle \mathbb{1}^i_{\mathbf{x}} \mathbb{1}^j_{\mathbf{y}}\big\rangle \leq \frac{1}{12}\sum_{n\geq 2} n^2 9^n (4\delta)^{n/2}$. Requiring this to be $< 1/28$ gives (2.2), provided that (2.17) and (2.18) are valid. Our next goal is to prove these estimates!

2.3 The universal projector

We first illustrate the estimate for $\langle P^{32}\rangle$, the ones for $\langle P^{34}\rangle$, ..., $\langle P^0\rangle$ being very similar, or simpler. We proceed as in [6, Section I.E and III]. Let $\{\psi_i\}_{i\geq 0}$ be an orthonormal set of eigenfunctions of H_Λ, numbered in order of increasing energy (i.e., E_0 is the ground state energy). Moreover, let ϕ be the classical state corresponding to the universal projection P^{32}, i.e., $P^{32} = |\phi\rangle\langle\phi|$. We write

$$\begin{aligned}
\big\langle P^{32}\big\rangle &= \frac{1}{Z_\Lambda}\sum_{i\geq 0} e^{-\beta E_i} |\langle\psi_i|\phi\rangle|^2 \qquad (2.20)\\
&= \frac{1}{Z_\Lambda}\Big[\sum_{i:\ E_i < E_0+\Delta|\Lambda|} e^{-\beta E_i}|\langle\psi_i|\phi\rangle|^2 + \sum_{i:\ E_i \geq E_0+\Delta|\Lambda|} e^{-\beta E_i}|\langle\psi_i|\phi\rangle|^2\Big]\\
&\equiv R_- + R_+,
\end{aligned}$$

where Δ is a cutoff, which can be fixed to be $\Delta = 1/16$: the criterion for the choice of Δ is that the energy $E_0 + \Delta|\Lambda|$ should be approximately in the middle between the ground state energy E_0 and the energy of ϕ. Neglecting ε, the ground state energy would be $-|\Lambda|/2$, while the energy of ϕ would be $-3|\Lambda|/8$, i.e., it would be separated from the ground state energy by a gap $2\Delta|\Lambda|$ with $\Delta = 1/16$.

2.3.1 Estimate of R_+

We start with the easier term, R_+. Using the condition $E_i \geq E_0 + \Delta|\Lambda|$ we get

$$R_+ \leq \frac{1}{Z_\Lambda} e^{-\beta(E_0+\Delta|\Lambda|)}. \tag{2.21}$$

To get a bound on Z_Λ, we restrict the trace to one of the (monomer-free) classical ground states, D_0. Thus,

$$Z_\Lambda \geq \left\langle D_0 | e^{\beta(U+\varepsilon T - zM)} | D_0 \right\rangle \geq e^{\beta\langle D_0|(U+\varepsilon T - zM)|D_0\rangle} = e^{\beta U_0} \tag{2.22}$$

with $U_0 = |\Lambda|/2$. Moreover, $T \leq U$, so that $H_\Lambda \geq -U(1+\varepsilon) + zM$ and, therefore, $E_0 \geq -U_0(1+\varepsilon)$. All in all we find

$$R_+ \leq e^{-\beta(\Delta|\Lambda| - \varepsilon U_0)} = e^{-\beta|\Lambda|(1/16 - \varepsilon/2)} \, . \tag{2.23}$$

2.3.2 Estimate of R_-

We now turn to the more subtle R_- term. Here we use the exponential localization strategy of [6]. We bound

$$R_- \leq \max_{i: E_i < E_0 + \Delta|\Lambda|} |\langle \psi_i | \phi \rangle|^2 \, . \tag{2.24}$$

The goal is to show that the overlap between the two states is exponentially small. It is clear that if ε were zero then ϕ and ψ_i would be orthogonal because they would belong to subspaces with very different energies. If $\varepsilon \neq 0$ is small, as is the case here, the two states are eigenfunctions of different but close Hamiltonians (with and without εT) with very different energies. The localization principle says that they are still almost orthogonal.

Let $A = -U + U_0 + zM$ and $B = -\varepsilon T$. We also need the projector Π onto the subspace of eigenvectors of A with eigenvalues larger than $\frac{3}{2}\Delta|\Lambda|$.

Note in particular that ϕ is in the range of Π. The eigenvalue equation for ψ_i reads

$$(A+B)\psi_i = (E_i + U_0)\psi_i \equiv \lambda\psi_i, \tag{2.25}$$

so that $\psi_i = -(A-\lambda+i\delta)^{-1}(B-i\delta)\psi_i$, for some $\delta \geq 0$. Therefore, $\langle\psi_i|\phi\rangle = -\langle\psi_i|(B-i\delta)(A-\lambda+i\delta)^{-1}\phi\rangle$ and, passing to the limit $\delta \to 0$,

$$\langle\psi_i|\phi\rangle = -\langle\psi_i|B(A-\lambda)^{-1}|\phi\rangle. \tag{2.26}$$

Not only ϕ is in the range of Π, but this is also the case for $B(A-\lambda)^{-1}\phi$. Therefore, in (2.26) we can freely add some projectors:

$$\langle\psi_i|\phi\rangle = -\langle\psi_i|\Pi B\Pi(A-\lambda)^{-1}|\phi\rangle. \tag{2.27}$$

We can now iterate this, as long as $(B(A-\lambda)^{-1})^n\phi$ is in the range of Π, which is the case if, e.g., $n \leq \Delta|\Lambda|/3$. We thus get

$$\langle\psi_i|\phi\rangle = (-1)^n\langle\psi_i|\big(\Pi B\Pi(A-\lambda)^{-1}\big)^n|\phi\rangle. \tag{2.28}$$

Now, $\|\Pi B\Pi\| \leq \varepsilon|\Lambda|$, and $\|\Pi(A-\lambda)^{-1}\| \leq (3\Delta|\Lambda|/2-\lambda)^{-1}$. Moreover, using the fact that $-T \leq U$, we have $H_\Lambda = -U + zM - \varepsilon T \leq -U(1-\varepsilon) + zM$. Therefore, if $z + 1/2 > \varepsilon/2$, we get $E_0 \leq -U_0(1-\varepsilon)$, so that,

$$\lambda = E_i + U_0 < E_0 + U_0 + \Delta|\Lambda| \leq \varepsilon U_0 + \Delta|\Lambda| = |\Lambda|(\varepsilon/2 + \Delta).$$

Consequently, $\|\Pi(A-\lambda)^{-1}\| \leq 2|\Lambda|^{-1}(\Delta-\varepsilon)^{-1}$. Using these estimates and (2.26) we obtain, choosing $n = \Delta|\Lambda|/3$,

$$|\langle\psi_i|\phi\rangle| \leq \Big(\frac{2\varepsilon}{\Delta-\varepsilon}\Big)^{\Delta|\Lambda|/3}, \tag{2.29}$$

which implies the desired exponential bound on R_-.

By inserting the bounds we obtained on R_+, R_- into (2.20), we get

$$\big\langle P^{32}\big\rangle^{2/|\Lambda|} \leq e^{-\beta(2\Delta-\varepsilon)} + \big(\frac{2\varepsilon}{\Delta-\varepsilon}\big)^{4\Delta/3}. \tag{2.30}$$

with $\Delta = 1/16$. By proceeding analogously, we find that $\langle P^{34}\rangle$ admits exactly the same bound.

The estimate for $\langle P^0\rangle$ is much simpler: in fact the state ϕ_0 corresponding to the projector P^0 is an eigenstate of H_Λ with energy $-U_0 + (z+1/2)|\Lambda|$, which together with (2.22) immediately implies (2.17).

Finally, we have to estimate $\langle P^{30} \rangle$ and $\langle P^{20} \rangle$. By using RP again we can bound both of them by $\langle P^0 \rangle^{1/2}$. This is accomplished by repeated reflections about the vertical planes separating regions occupied by dimers from regions occupied by monomers, see Fig.1. In this way, we obtain the inequalities $\langle P^{30} \rangle \leq \langle P^3 \rangle^{1/2} \langle P^0 \rangle^{1/2}$ and $\langle P^{20} \rangle \leq \langle P^2 \rangle^{1/2} \langle P^0 \rangle^{1/2}$, but $\langle P^3 \rangle$ and $\langle P^2 \rangle$ are smaller than 1, obviously. Putting things together we obtain the desired estimate (2.18), which concludes the proof of our main result.

3 Possible extensions and conclusion

Our model, consisting of dimers with (attractive) parallel dimer-dimer interaction and with a 'flipping' type 'kinetic energy', has been shown to have LRO at low temperature and not too large flipping rate. We can also allow monomers, as long as their chemical potential is not too negative.

Unlike other proofs of these properties, ours uses reflection positivity (RP), which has the advantage of simplification and the possibility of easily obtaining estimates of the allowed constants. It also has the advantage that the existence of RP allows one to make statements about the signs of some correlation functions and inequalities among them. We do not go into details about these matters in this paper, however. In the interest of simplicity we concentrated only on the existence of LRO.

Our work can easily be extended in several ways. One is to other dimensions. The dimers can be replaced by plaquettes in 3 dimensions, although they can also remain as dimers.

Another extension is to give a certain amount of dynamics to the monomers by adding a term to the Hamiltonian that annihilates a pair of adjacent monomers and replaces them with a dimer (and the reverse, of course). Other kinetic terms allowing, e.g., sliding of dimers over unoccupied sites, are not generally permitted because they break reflection positivity.

Extensions to other lattices are possible, too: for instance, the quantum dimer model on the hexagonal lattice is reflection positive. However, the proof of the chessboard estimate in that case is not as immediate as on the square lattice; see [4] for a use of repeated reflections on the hexagonal lattice in the context of the Peierls instability in graphene.

Acknowledgements. The research leading to these results has received funding from the European Research Council under the European Unions

Seventh Framework Programme ERC Starting Grant CoMBoS, grant agreement n^o 239694 (A.G.) and from U.S. National Science Foundation grant PHY-1265118 (E.H.L.).

References

[1] C. Borgs, R. Kotecky, D. Ueltschi: Low temperature phase diagrams for quantum perturbations of classical spin systems, Comm. Math. Phys. **181** 409-446 (1996).

[2] N. Datta, R. Fernandez, J. Fröhlich: Low-temperature phase diagrams of quantum lattice systems. I. Stability for quantum perturbations of classical systems with finitely-many ground states, J. Stat. Phys. **84**, 455-534 (1996).

[3] F. J. Dyson, E. H. Lieb, B. Simon: Phase Transitions in Quantum Spin Systems with Isotropic and Nonisotropic Interactions, J. Stat. Phys. **18**, 335-383 (1978).

[4] R. Frank, E. H. Lieb: Possible Lattice Distortions in the Hubbard Model for Graphene, Phys. Rev. Lett. **107**, 066801 (2011).

[5] J. Fröhlich, R. Israel, E. H. Lieb, B. Simon: Phase Transitions and Reflection Positivity. I. General Theory and Long Range Lattice Models, Comm. Math. Phys. **62**, 1-34 (1978).

[6] J. Fröhlich, E. H. Lieb: Phase transitions in anisotropic lattice spin systems, Comm. Math. Phys. **60**, 233-267 (1978).

[7] J. Fröhlich, B. Simon, T. Spencer: Infrared bounds, phase transitions, and continuous symmetry breaking, Comm. Math. Phys. **50**, 79-85 (1976).

[8] A. Giuliani, V. Mastropietro, F. Toninelli: Height fluctuations in interacting dimers, arXiv:1406.7710.

[9] A. Giuliani, V. Mastropietro, F. Toninelli: Height fluctuations in non-integrable classical dimers, Europhys. Lett. **109**, 60004 (2015).

[10] O. J. Heilmann: The use of reflection as symmetry operation in connection with Peierls' argument, Comm. Math. Phys. **36**, 91-114 (1974).

[11] O. J. Heilmann, E. Praestgaard: Crystalline ordering in lattice models of hard rods with nearest neighbour attraction, Chem. Phys. **24**, 119-123 (1977)

[12] O. J. Heilmann, E. Praestgaard: Phase transition in a lattice gas with third nearest neighbour exclusion on a square lattice, J. Phys. A **7**, 1913-1917 (1974).

[13] R. Moessner and K. S. Raman: Quantum dimer models, in *Introduction to frustrated magnetism*, Springer Series in Solid-State Sciences **164**, 437-479 (2011).

[14] K. Osterwalder, R. Schrader: Axioms for Euclidean Green's functions, I, Comm. Math. Phys. **31**, 83-112 (1973); and II, *ibid* **42**, 281-305 (1975).

[15] S.A. Pirogov, Ya.G. Sinai: Phase diagrams of classical lattice systems, I, Theor. Math. Phys. **25**, 358-369 (1975); and II, *ibid* **26**, 61-76 (1976).

[16] A. Ralko, D. Poilblanc, R. Moessner: Generic Mixed Columnar-Plaquette Phases in Rokhsar-Kivelson Models, Phys. Rev. Lett. **100**, 037201 (2008).

[17] D. S. Rokhsar, S. A. Kivelson: Superconductivity and the Quantum Hard-Core Dimer Gas, Phys. Rev. Lett. **61**, 2376 (1988).